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The gauge algebra of double field theory and Courant brackets

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Abstract: We investigate the symmetry algebra of the recently proposed field theory on a doubled torus that describes closed string modes on a torus with both momentum and winding. The gauge parameters are constrained fields on the doubled space and transform as vectors under T-duality. The gauge algebra defines a T-duality covariant bracket. For the case in which the parameters and fields are T-dual to ones that have momentum but no winding, we find the gauge transformations to all orders and show that the gauge algebra reduces to one obtained by Siegel. We show that the bracket for such restricted parameters is the Courant bracket. We explain how these algebras are realised as symmetries despite the failure of the Jacobi identity.

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1 Introduction and main results

The closed string on a D-dimensional torus background can be formulated in terms of an infinite set of fields that are in fact fields on the doubled torus parameterized by D spacetime coordinates x^i and D additional coordinates $\tilde{x}_i$ dual to winding.[1] In a recent paper [1] we began a detailed investigation of a *double* field theory. We focused on the sector of closed string theory consisting of fields $e_{ij}(x,\tilde{x}) \equiv h_{ij}(x,\tilde{x}) + b_{ij}(x,\tilde{x})$ and the dilaton $d(x,\tilde{x})$. Here h_{ij} and b_{ij} are $D \times D$ matrices depending on the $2D$ coordinates $(x^i, \tilde{x}_i)$ and they represent doubled gravity and antisymmetric tensor fluctuations around constant backgrounds $E_{ij} \equiv G_{ij} + B_{ij}$. A T-duality invariant, gauge invariant double field theory was constructed to cubic order in the fields. A full construction to all orders remains a major challenge; if achieved, the resulting theory would likely be a consistent truncation of the complete closed string theory. Our construction relied on the formulation [2] of closed string field theory on tori. Earlier work on double field theory includes that of Tseytlin [3] and that of Siegel [4].

[1] As explained in later sections, our notation covers the case in which there are a number of non-compact dimensions. The absence of winding in the non-compact directions requires that all fields are independent of the corresponding $\tilde{x}$'s. For simplicity, in the introduction we focus on the case with all dimensions compact.

A key element in the construction of [1] was the constraint that all fields and gauge parameters must be annihilated by the operator Δ given by

$$\Delta = -\,2\,\frac{\partial}{\partial x^i}\frac{\partial}{\partial \tilde{x}_i} = -\,2\,\partial_i\tilde{\partial}^i\,, \tag{1.1}$$

having set $\alpha' = 1$. The constraint $\Delta = 0$ is needed for gauge invariance and consistency. It is the field theory version of the constraint $L_0 - \bar{L}_0 = 0$ of closed string field theory [5]. Since Δ is a second-order differential operator, the product of two fields in the kernel of Δ need not be in the kernel of Δ. This means that a projection to the kernel of Δ is necessary in the products that appear in the action and gauge transformations of the double field theory [1]. These projectors and certain cocycle factors make the double field theory non-standard and novel.

The theory simplifies considerably if the fields and gauge parameters are restricted to be independent of $\tilde{x}_i$, as the constraint $\Delta = 0$ is then automatically satisfied. No projectors or cocycles are needed and it reduces to a conventional field theory of fields depending only on x^i. Similarly, a conventional field theory arises for fields and parameters that depend just on $\tilde{x}_i$, or just on the coordinates of any D-dimensional toroidal subspace N of the double space that is totally null with respect to the signature (D, D) metric

$$ds^2 = 2dx^i d\tilde{x}_i\,. \tag{1.2}$$

T-duality transforms any such totally null or maximally isotropic subspace into another. We will refer to fields with dependence only on a null subspace as *restricted* fields, and in so doing we will always require all fields and parameters to be restricted to the *same* space N. Any fields constructed with products of restricted fields are automatically restricted. Restricted fields and their products satisfy $\Delta = 0$. The converse is not true: a set of fields in the kernel of Δ need not be restricted because products of the fields need not be in the kernel of Δ. The reduction to restricted fields — discussed in detail in section 5 — is thus a significant truncation of the field space of double field theory.

String theory on a D-torus has an $O(D, D; \mathbb{Z})$ T-duality symmetry and double field theory inherits this T-duality symmetry. In fact, the double field theory would have a formal $O(D, D)$ symmetry if all dimensions were non-compact. We formulate double field theory in a way that is largely independent of the number of dimensions that are toroidal, and we will find it convenient to refer to expressions as being $O(D, D)$ covariant (or invariant) when we in fact mean covariant (or invariant) under the subgroup of $O(D, D)$ preserving the boundary conditions on periodic coordinates, together with the condition of no winding in non-compact directions. A central role will be played by the requirement that the gauge algebra have such $O(D, D)$ covariance.

In this paper we focus on the gauge transformations and their algebra. We find here the complete non-linear transformations for restricted fields and the corresponding gauge algebra. Our results are $O(D, D)$ covariant, so that they apply for any choice of null subspace for the restricted fields. The constraint to restricted fields was also assumed by Siegel [4]. Using a set of fields larger than the one used here, as well as additional gauge invariances, he proposed a gauge and duality invariant action to all orders in the fields.

Most of our work in this paper relates to restricted fields, so comparison with the results of [4] will feature at various points. In particular, we show that our gauge algebra reduces to that of Siegel when the gauge parameters are restricted.

To first order in the fields, the gauge transformations of $e_{ij}(x,\tilde{x})$ and $d(x,\tilde{x})$ take the form [1]

$$\begin{aligned}\delta_\lambda e_{ij} &= D_i\bar{\lambda}_j + \bar{D}_j\lambda_i + \frac{1}{2}(\lambda\cdot D + \bar{\lambda}\cdot\bar{D})e_{ij} + \frac{1}{2}(D_i\lambda^k - D^k\lambda_i)e_{kj} - e_{ik}\frac{1}{2}(\bar{D}^k\bar{\lambda}_j - \bar{D}_j\bar{\lambda}^k)\,,\\ \delta_\lambda d &= -\frac{1}{4}(D\cdot\lambda + \bar{D}\cdot\bar{\lambda}) + \frac{1}{2}(\lambda\cdot D + \bar{\lambda}\cdot\bar{D})\,d\ .\end{aligned} \tag{1.3}$$

Here $\lambda_i(x,\tilde{x})$ and $\bar{\lambda}_i(x,\tilde{x})$ are independent real gauge parameters, and the derivatives $D, \bar{D}$ are defined by

$$D_i = \partial_i - E_{ik}\,\tilde{\partial}^k\,, \qquad \bar{D}_i = \partial_i + E_{ki}\,\tilde{\partial}^k\,. \tag{1.4}$$

They are independent real derivatives with respect to right- and left-moving coordinates [1]. Indices are raised and lowered with the background metric G_{ij} and $a\cdot b = G^{ij}a_ib_j$. It is straightforward to verify that $D^2 - \bar{D}^2 = 2\Delta$. The above gauge transformations are reducible. If we take

$$\lambda_i = D_i\chi, \qquad \bar{\lambda}_i = -\bar{D}_i\chi\,, \tag{1.5}$$

for arbitrary $\chi(x,\tilde{x})$, then the fields are left invariant. Therefore, $\lambda_i \to \lambda_i + D_i\chi$ and $\bar{\lambda}_i \to \bar{\lambda}_i - \bar{D}_i\chi$ constitutes a "symmetry for a symmetry." This will play an important role in our discussion.

The gauge algebra is $[\delta_{\lambda_1}, \delta_{\lambda_2}] = \delta_{\lambda_{12}} + \cdots$, where to leading nontrivial order

$$\begin{aligned}\lambda_{12}^i &= \frac{1}{2}(\lambda_2\cdot D + \bar{\lambda}_2\cdot\bar{D})\,\lambda_1^i - \frac{1}{4}\Big[\,\lambda_2\cdot D^i\lambda_1 - \bar{\lambda}_2\cdot D^i\bar{\lambda}_1\Big] - (1\leftrightarrow 2)\,,\\ \bar{\lambda}_{12}^i &= \frac{1}{2}(\lambda_2\cdot D + \bar{\lambda}_2\cdot\bar{D})\,\bar{\lambda}_1^i + \frac{1}{4}\Big[\,\lambda_2\cdot\bar{D}^i\lambda_1 - \bar{\lambda}_2\cdot\bar{D}^i\bar{\lambda}_1\Big] - (1\leftrightarrow 2)\ .\end{aligned} \tag{1.6}$$

A projection to the kernel of Δ is necessary (and implicit) in the products that appear in (1.3) and in (1.6). For restricted fields and parameters, however, the projections are not needed as the $\Delta = 0$ constraint is automatically satisfied. The gauge algebra (1.6) defines a bracket of two gauge parameters and this bracket does not satisfy the Jacobi identity [1].

The results above for the gauge transformations and algebra respect certain rules of index contraction that arise from the string theory and result in $O(D,D)$ or T-duality covariance of the double field theory. While we display just one kind of index $i, j, k, \ldots$, some indices should be thought as barred and some as unbarred. On the field e_{ij} the first index, i, is unbarred and the second, j, is barred. On $\bar{\lambda}_i$ and $\bar{D}_i$ the index is barred. On λ_i and D_i the index is unbarred. Contractions can only occur between indices of the same type, and equations must relate objects with identical index structure. The metric G_{ij} used to contract indices can be viewed as having two barred or two unbarred indices.[2]

[2]These rules of contraction can be understood using $O(D)_L \times O(D)_R$ indices, with unbarred indices for $O(D)_L$ and barred ones for $O(D)_R$.

We can ask if the gauge transformations and gauge algebra displayed above can be extended to all orders for restricted fields. We answer this question in the affirmative in section 2. We show that the only non-linear correction is the addition to $\delta_\lambda e_{ij}$ of a term quadratic in e_{ij}. The result is given in (2.20). The gauge transformations then close off-shell for restricted fields. In fact, the full algebra remains that of (1.6) with (field independent) structure constants. The full gauge transformations remain reducible, and parameters of the form (1.5) still leave the fields invariant. We show that our gauge transformations are related to the standard ones for a metric and B-field via field redefinitions. We stress that the algebra closes without use of the equations of motion, but with repeated use of the condition that the fields and parameters are restricted.

To clarify the meaning of the gauge transformations, in section 3 we rewrite the gauge parameters $\lambda_i, \bar{\lambda}_i$ in terms of quantities ξ^i, $\tilde{\xi}_i$, and use $O(D,D)$ covariant notation, defining

$$\partial_M \equiv \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix} \qquad \Sigma^M \equiv \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix} , \quad \eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} . \tag{1.7}$$

The familiar transformations arise for parameters that are independent of $\tilde{x}$, so that the gauge parameters $\xi^i(x)$ and $\tilde{\xi}_i(x)$ are a vector field and a one-form over the spacetime M with coordinates x^i. These are related to the parameters for infinitesimal diffeomorphisms of M and B-field gauge transformations respectively [1].

The gauge algebra (1.6) for general $\Sigma(x,\tilde{x})$ can be rewritten as $[\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{\Sigma_{12}}$ where $\Sigma_{12} = -[\Sigma_1, \Sigma_2]_C$ and the C-bracket is defined by

$$[\Sigma_1, \Sigma_2]_C \equiv \Sigma_{[1}^N \partial_N \Sigma_{2]}^M - \frac{1}{2}\eta^{MN}\eta_{PQ}\Sigma_{[1}^P \partial_N \Sigma_{2]}^Q . \tag{1.8}$$

We use the convention $U_{[r}V_{s]} = U_rV_s - U_sV_r$. The first term in the right-hand side of (1.8) is the Lie bracket $[\Sigma_1, \Sigma_2] = \Sigma_{[1}^N \partial_N \Sigma_{2]}^M$ while the second is unexpected. In this form, the gauge algebra coincides with that derived by Siegel [4] (where the derivative defined using the C-bracket was called a 'new' Lie derivative). The C-bracket is manifestly $O(D,D)$ covariant. With $O(D,D)$ notation, the transformations (1.5) that do not act on the fields have parameters that take the form

$$\Sigma^M = \eta^{MN}\partial_N \chi . \tag{1.9}$$

Remarkably, the C-bracket that arises in the gauge algebra here is related to brackets that have been prominent in the mathematics literature. In section 7 we show that for parameters restricted to be independent of $\tilde{x}_i$, the C-bracket is precisely the Courant bracket [6], a central construction in generalised geometry [7–9]. Indeed, $\tilde{x}$-independent gauge parameters $\xi^i(x)$ and $\tilde{\xi}_i(x)$ together give a section of the formal sum $(T \oplus T^*)M$ of tangent and cotangent bundles and the Courant bracket is defined on such sections. Parameters restricted to an arbitrary null N can be regarded as sections of $(T \oplus T^*)N$ and the C-bracket becomes the Courant bracket on $(T \oplus T^*)N$. The choice of N breaks the $O(D,D)$ covariance of the C-bracket. Since the choice of N need not be made explicit, the C-bracket can be regarded as an $O(D,D)$ covariantization of the Courant bracket. In section 8 we show that it is equivalent (for restricted parameters) to the Courant-like bracket of [9] that treats vectors and one-forms symmetrically.

Neither the C-bracket nor the Courant bracket satisfy the Jacobi identity. It is then natural to ask how this failure of the Jacobi identity can be consistent with the realisation of these brackets in a symmetry algebra. To answer this question we consider the associated infinitesimal field transformations δ_Σ. The commutator of two transformations acting on fields gives

$$[\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1,\Sigma_2]}\,. \tag{1.10}$$

Here $[\Sigma_1\,,\Sigma_2]$ is the bracket of gauge parameters, which for our case is (-1 times) the C-bracket. The bracket has a non-vanishing Jacobiator J, defined by

$$J(\Sigma_1, \Sigma_2, \Sigma_3) \equiv [\,[\Sigma_1, \Sigma_2]\,, \Sigma_3\,] + [\,[\Sigma_2, \Sigma_3]\,, \Sigma_1\,] + [\,[\Sigma_3, \Sigma_1]\,, \Sigma_2\,]\,. \tag{1.11}$$

The commutators of transformations automatically satisfy

$$[\,[\delta_{\Sigma_1}, \delta_{\Sigma_2}]\,, \delta_{\Sigma_3}\,] + [\,[\delta_{\Sigma_2}, \delta_{\Sigma_3}]\,, \delta_{\Sigma_1}\,] + [\,[\delta_{\Sigma_3}, \delta_{\Sigma_1}]\,, \delta_{\Sigma_2}\,] = 0\,, \tag{1.12}$$

when acting on fields. The left hand side, however, can be evaluated using (1.10) to give $\delta_{J(\Sigma_1,\Sigma_2,\Sigma_3)}$ which can only be consistent with the above condition if $\delta_{J(\Sigma_1,\Sigma_2,\Sigma_3)}$ is zero when acting on fields. This requires that the Jacobiator $J(\Sigma_1, \Sigma_2, \Sigma_3)$ be a parameter of the form (1.9) for a trivial gauge transformation leaving all fields invariant. As we will discuss, the Jacobiators for the C-bracket and the Courant bracket are precisely of this form, so that the algebra can indeed be consistently realised on fields.

In a theory with redundant gauge symmetry, the gauge algebra is ambiguous since the bracket of two gauge parameters can be changed by adding a parameter that generates a trivial symmetry. If σ_{12} is any nonvanishing gauge parameter of the form (1.9) so that $\delta_{\sigma_{12}}$ leaves all fields invariant, then (1.10) can be changed to

$$[\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1,\Sigma_2]+\sigma_{12}}\,. \tag{1.13}$$

We shall argue that no such σ_{12} can be constructed from Σ_1 and Σ_2 in a way that is $O(D, D)$ covariant. It is an important point in this paper that the duality symmetry $O(D, D)$ fixes this ambiguity and the C-bracket cannot be changed while preserving $O(D, D)$ covariance. This ambiguity, however, plays a useful role in relating our results to those for conventional field theory, which are not $O(D, D)$ covariant. The Jacobiator and redundant symmetries will be discussed further in section 6.

2 The gauge transformations

Our aim in this section is to investigate the higher order corrections to the gauge transformations and algebra reviewed in the introduction. We assume fields and gauge parameters restricted to some isotropic subspace N. For such fields no projection to the kernel of Δ is needed and the cocycles vanish, so that the calculations are those for a conventional classical field theory. We work in an $O(D, D)$ covariant framework so that the result applies for *any* choice of isotropic subspace N. We find the full non-linear transformations and algebra for restricted fields.

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Before proceeding, we briefly discuss our notation; see [1] for further details. The simplest case is that in which all D coordinates x^i are periodic, but our notation also covers the case in which there are n non-compact dimensions. Then $x^i = (x^\mu, x^a)$ split into coordinates x^μ on the n-dimensional Minkoswski space $\mathbb{R}^{n-1,1}$ and coordinates x^a on the d-torus T^d where $n+d = D$. The absence of winding in the x^μ directions requires that all fields and gauge parameters are independent of $\tilde{x}_\mu$ so that $\partial/\partial\tilde{x}_\mu = 0$ on all fields and parameters. The fields then depend only on $x^\mu, x^a, \tilde{x}_a$. T-duality is the group $O(d, d; \mathbb{Z})$ acting on the doubled torus with coordinates $x^a, \tilde{x}_a$. For restricted fields, the totally null subspace N has coordinates x^μ together with a totally null d-dimensional torus subspace of the torus $(x^a, \tilde{x}_a)$, e.g. (x^μ, x^a) or $(x^\mu, \tilde{x}_a)$. Such spaces N are related by the action of $O(d, d; \mathbb{Z})$.

From [1] and the discussion in the introduction, the full form of the gauge transformations should have the following properties for restricted fields:

- The gauge algebra closes.
- The transformations are $O(D, D)$ covariant.
- All index contractions in the transformations should be of the allowed kind of barred indices with barred indices or of unbarred indices with unbarred indices. This is necessary for $O(D, D)$ covariance.
- For any choice of isotropic subspace N, the transformations should be related to the standard Einstein plus B-field transformations by redefinitions of the fields and parameters.

We start by investigating the last two criteria. These will be sufficient to find the full non-linear form of the gauge transformations for restricted fields and we will then check the algebra of these transformations in an appendix. Consider the restriction to fields that have no $\tilde{x}$ dependence, so that setting $D = \bar{D} = \partial$ brings the transformations (1.3) to a form that can be related to the standard gauge transformations. In the standard Einstein plus B-field theory, the gauge transformations are the diffeomorphisms with infinitesimal parameter ϵ^i and antisymmetric tensor gauge transformations with infinitesimal parameter $\tilde{\epsilon}_i$. The full metric is $G_{ij}+h_{ij}$ and the antisymmetric tensor gauge field is $B_{ij}+b_{ij}$ where G_{ij} and B_{ij} are constant background fields. For the combined fluctuation field $\check{e}_{ij} = h_{ij} + b_{ij}$, the transformations are

$$\delta\check{e}_{ij} = \delta^{(0)}\check{e}_{ij} + \delta^{(1)}\check{e}_{ij} \,, \tag{2.1}$$

with

$$\begin{aligned} \delta^{(0)}\check{e}_{ij} &= \partial_i\epsilon_j + \partial_j\epsilon_i - (\partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i) \,, \\ \delta^{(1)}\check{e}_{ij} &= \epsilon^k\partial_k\check{e}_{ij} + (\partial_i\epsilon^k)\,\check{e}_{kj} + \check{e}_{ik}\,(\partial_j\epsilon^k) \,. \end{aligned} \tag{2.2}$$

Here and in what follows, indices i, j are raised and lowered using the background metric G_{ij}.

In order to connect with our formalism we rewrite ϵ and $\bar{\epsilon}$ in terms of λ and $\bar{\lambda}$:

$$\epsilon_i = \frac{1}{2}(\lambda_i + \bar{\lambda}_i) \,, \quad \tilde{\epsilon}_i = \frac{1}{2}(\lambda_i - \bar{\lambda}_i) \,. \tag{2.3}$$

In [1] it was shown that the field $\breve{e}_{ij}$ is related to (e_{ij}, d) by $\breve{e}_{ij} = f_{ij}(e,d)$, where

$$f_{ij}(e,d) = e_{ij} + \frac{1}{2} e_i{}^k e_{kj} + \text{cubic corrections}\,, \tag{2.4}$$

and this maps the transformations (2.2) to (1.3), up to terms quadratic in fields. The redefinition takes the full non-linear transformations of $\breve{e}_{ij}$ given in (2.2) to transformations of e_{ij}, and the condition that there should only be allowed contractions in the transformation of e places stringent constraints on f. The redefinition $\breve{e}_{ij} = f_{ij}(e,d)$ gives terms quadratic in the fields that include ones with disallowed contractions. These 'bad terms' can be eliminated by taking

$$f_{ij}(e,d) = e_{ij} + \frac{1}{2}\, e_i{}^k e_{kj} + \frac{1}{4} e_i{}^k e_{kl}\, e^l{}_j + \text{quartic corrections}\,. \tag{2.5}$$

This is easily extended to arbitrary order, and one soon finds that requiring only allowed contractions fixes f to be

$$f = \left(1 - \frac{1}{2}\, e\right)^{-1} e\,, \tag{2.6}$$

so that

$$\breve{e} = \left(1 - \frac{1}{2}\, e\right)^{-1} e\,, \tag{2.7}$$

where we use matrix notation, so that the first few terms are as in (2.5). The function (2.6) first arose in the work of Michishita [10].[3] We now show that this gives no bad contractions and use this to find the full non-linear gauge transformation of e_{ij}.

It is an immediate consequence of the definition (2.6) that $\breve{e}$ and e commute:

$$\breve{e}\, e = e\, \breve{e}\,. \tag{2.8}$$

It follows that

$$e = \left(1 - \frac{1}{2}\, e\right) \breve{e} = \breve{e} \left(1 - \frac{1}{2}\, e\right). \tag{2.9}$$

The above leads to

$$\breve{e} - e = \frac{1}{2}\, e\breve{e} = \frac{1}{2} \breve{e} e\,, \tag{2.10}$$

and one readily verifies that

$$\left(1 + \frac{1}{2}\, \breve{e}\right) \left(1 - \frac{1}{2}\, e\right) = 1\,. \tag{2.11}$$

Finally, varying (2.10) and using (2.11) we find a relation between arbitrary variations,

$$\delta e = \left(1 - \frac{1}{2}\, e\right) \delta\breve{e} \left(1 - \frac{1}{2}\, e\right). \tag{2.12}$$

[3] This function was proposed in [10] as a simple ansatz for the relation between the conventional variable $\breve{e}$ and the string field variable that satisfies a number of constraints but is not uniquely selected. It was shown in [1] section 4.5, that the relation between $\breve{e}$ and the string field variable has explicit dependence on d.

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The standard gauge transformations (2.2) can be rearranged using (2.3) to give

$$\begin{aligned}
\delta^{(0)}\breve{e}_{ij} &= \partial_i\bar{\lambda}_j + \partial_j\lambda_i\,, \\
\delta^{(1)}\breve{e}_{ij} &= \frac{1}{2}\Big[(\lambda+\bar{\lambda})\cdot\partial\Big]\breve{e}_{ij} + \left[\frac{1}{2}(\delta^{(0)}\breve{e}_i{}^k) + \mathcal{N}_i{}^k\right]\breve{e}_{kj} + \breve{e}_{ik}\left[\frac{1}{2}(\delta^{(0)}\breve{e}^k{}_j) - \bar{\mathcal{N}}^k{}_j\right],
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
\mathcal{N}_i{}^k &= \partial_i\lambda^k - \partial^k\lambda_i\,, \\
\bar{\mathcal{N}}^k{}_j &= \partial^k\bar{\lambda}_j - \partial_j\bar{\lambda}^k\,.
\end{aligned} \tag{2.14}$$

Note that $\delta^{(0)}\breve{e}_{ij} = \delta^{(0)}e_{ij}$ where $\delta^{(0)}e_{ij}$ is the zeroth order transformation of e_{ij} in (1.3). Using matrix notation, the full gauge transformation can be written as

$$\delta\breve{e} = \delta^{(0)}e + \frac{1}{2}\Big[(\lambda+\bar{\lambda})\cdot\partial\Big]\breve{e} + \left[\frac{1}{2}\delta^{(0)}e + \mathcal{N}\right]\breve{e} + \breve{e}\left[\frac{1}{2}\delta^{(0)}e - \bar{\mathcal{N}}\right]. \tag{2.15}$$

We now determine δe using (2.12). Using (2.9) and (2.12) we find

$$\begin{aligned}
\delta e = \delta^{(0)}e &- \frac{1}{2}e\,\delta^{(0)}e - \frac{1}{2}(\delta^{(0)}e)e + \frac{1}{4}e(\delta^{(0)}e)e + \frac{1}{2}\Big[(\lambda+\bar{\lambda})\cdot\partial\Big]e \\
&+ \left(1-\frac{1}{2}e\right)\left[\frac{1}{2}\delta^{(0)}e + \mathcal{N}\right]e + e\left[\frac{1}{2}\delta^{(0)}e - \bar{\mathcal{N}}\right]\left(1-\frac{1}{2}e\right).
\end{aligned} \tag{2.16}$$

Expanding out and cancelling some terms we find

$$\delta e = \delta^{(0)}e + \frac{1}{2}\Big[(\lambda+\bar{\lambda})\cdot\partial\Big]e + \mathcal{N}e - e\bar{\mathcal{N}} - \frac{1}{4}\,e\,(\delta^{(0)}e)^t\,e\,, \tag{2.17}$$

where we made use of the identity

$$\frac{1}{2}\,\delta^{(0)}e + \mathcal{N} - \bar{\mathcal{N}} = \frac{1}{2}\,(\delta^{(0)}e)^t\,. \tag{2.18}$$

Restoring explicit indices in (2.17) we obtain

$$\begin{aligned}
\delta_\lambda e_{ij} =&\,\partial_i\bar{\lambda}_j + \partial_j\lambda_i \\
&+ \frac{1}{2}\,(\lambda^k + \bar{\lambda}^k)\partial_k\,e_{ij} + \frac{1}{2}\,(\partial_i\lambda^k - \partial^k\lambda_i)\,e_{kj} - e_{ik}\,\frac{1}{2}(\partial^k\bar{\lambda}_j - \partial_j\bar{\lambda}^k) \\
&- \frac{1}{4}\,e_{ik}\,(\partial^l\bar{\lambda}^k + \partial^k\lambda^l)\,e_{lj}\,.
\end{aligned} \tag{2.19}$$

This is the full non-linear transformation for fields that are independent of $\tilde{x}$ and indeed has no bad contractions. For general polarisations, some of the derivatives ∂_i should become D_i and some should become $\bar{D}_i$. There is a unique way of doing this which uses only allowed contractions:

$$\begin{aligned}
\delta_\lambda e_{ij} =&\,D_i\bar{\lambda}_j + \bar{D}_j\lambda_i \\
&+ \frac{1}{2}\,(\lambda\cdot D + \bar{\lambda}\cdot\bar{D})e_{ij} + \frac{1}{2}\,(D_i\lambda^k - D^k\lambda_i)\,e_{kj} - e_{ik}\,\frac{1}{2}(\bar{D}^k\bar{\lambda}_j - \bar{D}_j\bar{\lambda}^k) \\
&- \frac{1}{4}\,e_{ik}\,(D^l\bar{\lambda}^k + \bar{D}^k\lambda^l)\,e_{lj}\,.
\end{aligned} \tag{2.20}$$

This is our final answer for the gauge transformations. Note that the $\delta^{(1)}$ transformations derived in [1] and cited in the introduction are correctly generated. It is remarkable that only one extra term quadratic in fields is needed, so that the transformations are polynomial. Since the gauge algebra of the standard transformations closes, the transformations we obtained from these by field redefinitions should also have a closed algebra. In fact they do, as we have confirmed explicitly by direct computation (details in the appendix), and the gauge algebra is precisely (1.6). Note that this algebra has structure constants, not field-dependent structure functions.

The gauge transformation (1.3) of the dilaton satisfies the gauge algebra (1.6) and only involves good contractions. For restricted fields, these transformations can be obtained from those of the scalar dilaton ϕ and string-frame metric g_{ij} by the field relation $e^{-2d} = e^{-2\phi}\sqrt{-g}$, as discussed in [1]. Then we can take (1.3) as the full transformations of d exact to all orders in the fields (for restricted fields). This can be thought of as fixing the field-redefinition ambiguity. It is straightforward to check that gauge transformations with gauge parameters (1.5) leave both e_{ij} and d invariant so that, as expected, the gauge symmetry is still reducible. It remains to discuss the $O(D,D)$ covariance of the transformations and algebra. For this it is convenient to streamline the notation, as we do next.

3 O(D,D) rewriting of the gauge algebra

In this section we rewrite the gauge algebra (1.6) in a simpler form, using $O(D,D)$ covariant notation and the formalism introduced in [1]. As a first step, we define ξ^i and $\tilde{\xi}_i$ in terms of the gauge parameters λ and $\bar{\lambda}$:

$$\xi^i \equiv \frac{1}{2}(\lambda^i + \bar{\lambda}^i), \qquad \tilde{\xi}_i \equiv \frac{1}{2}(-E_{ji}\lambda^j + E_{ij}\bar{\lambda}^j)\,. \tag{3.1}$$

For reference, we also record the inverse relations:

$$\lambda_i = -\tilde{\xi}_i + E_{ij}\xi^j\,, \qquad \bar{\lambda}_i = \tilde{\xi}_i + E_{ji}\,\xi^j\,. \tag{3.2}$$

In the above indices are raised and lowered with the background metric G_{ij}. As we noted in the introduction, the partial derivatives $(\partial_i, \tilde{\partial}^i)$ with respect to the coordinates $(x^i, \tilde{x}_i)$ are related to the derivatives $(D_i, \bar{D}_i)$ by

$$D_i = \partial_i - E_{ik}\,\tilde{\partial}^k\,, \qquad \bar{D}_i = \partial_i + E_{ki}\,\tilde{\partial}^k\,, \tag{3.3}$$

with the inverse relations

$$\tilde{\partial}^i = \frac{1}{2}(-D^i + \bar{D}^i)\,, \qquad \partial_i = \frac{1}{2}(E_{ji}D^j + E_{ij}\bar{D}^j)\,. \tag{3.4}$$

It is then straightforward to verify that

$$\frac{1}{2}(\lambda^i D_i + \bar{\lambda}^i \bar{D}_i) = \xi^i\partial_i + \tilde{\xi}_i\tilde{\partial}^i\,. \tag{3.5}$$

Following [1], we can combine x and $\tilde{x}$ coordinates, ∂ and $\tilde{\partial}$ derivatives, and ξ and $\tilde{\xi}$ parameters into $O(D,D)$ covariant expressions

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}, \qquad \partial_M \equiv \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix}, \qquad \Sigma^M \equiv \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix}. \tag{3.6}$$

Here $M = 1,\ldots,2D$. The original space M has coordinates x^i and the dual space $\tilde{M}$ has coordinates $\tilde{x}_i$. Together these combine to form the doubled space $\hat{M} = M \times \tilde{M}$ with coordinates X^M. The parameters $\xi^i(x,\tilde{x})$ and $\tilde{\xi}_i(x,\tilde{x})$ have been combined to form $\Sigma^M(X)$. Note that with these definitions the transport derivative takes the form $\xi^i\partial_i + \tilde{\xi}_i\tilde{\partial}^i = \Sigma^M\partial_M$. In this basis the metric η_{MN} is given by

$$\eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.7}$$

We use this metric to raise and lower indices. We therefore have

$$\partial^M = \begin{pmatrix} \partial_i \\ \tilde{\partial}^i \end{pmatrix} \qquad \Sigma_M = \begin{pmatrix} \xi^i \\ \tilde{\xi}_i \end{pmatrix}. \tag{3.8}$$

We also note that

$$\Delta = -\eta^{MN}\partial_M\partial_N = -\,\partial^M\partial_M = -2\partial_i\,\tilde{\partial}^i\,, \tag{3.9}$$

and therefore fields A, B restricted to an arbitrary isotropic subspace N satisfy

$$\partial^M\partial_M A = \partial^M\partial_M B = 0\,, \quad (\partial_M A)(\partial^M B) = 0\,. \tag{3.10}$$

Finally, a short calculation then shows that for any scalar operator $\mathcal{O}$

$$-\frac{1}{4}\Big[\lambda_2\cdot\mathcal{O}\lambda_1 - \bar{\lambda}_2\cdot\mathcal{O}\,\bar{\lambda}_1\Big] = \frac{1}{2}\,\Sigma_2^M\,\mathcal{O}\,\Sigma_{1M}\,. \tag{3.11}$$

The $2D$-component vectors Σ^M, ∂^M, and X^M all transform under $O(D,D;\mathbb{Z})$ by the action of the integer-valued $2D\times 2D$ matrix g

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad g^t\eta g = \eta\,. \tag{3.12}$$

The C-bracket is covariant under this action of $O(D,D;\mathbb{Z})$. If all dimensions are non-compact so that $\hat{M} = \mathbb{R}^{2D}$, then the continuous group $O(D,D)$ is a symmetry of the C-bracket while if some of the dimensions are compact, $M = \mathbb{R}^n\times T^d$, the symmetry is broken to the subgroup $O(n,n)\times O(d,d;\mathbb{Z})$ preserving the periodicities of the coordinates.

We now use these relations to rewrite the algebra in an $O(D,D)$ covariant way. With the help of (3.5) and (3.11) the gauge algebra (1.6) can be rewritten as

$$\begin{aligned} \lambda_{12}^i &= \Sigma_2^N\partial_N\lambda_1^i + \frac{1}{2}\,\Sigma_2^N\,D^i\,\Sigma_{1N} \;-\; (1\leftrightarrow 2)\,, \\ \bar{\lambda}_{12}^i &= \Sigma_2^N\partial_N\,\bar{\lambda}_1^i - \frac{1}{2}\,\Sigma_2^N\,\bar{D}^i\Sigma_{1N} \;-\; (1\leftrightarrow 2)\,. \end{aligned} \tag{3.13}$$

Using (3.1) to define parameters Σ_{12} and $\tilde{\Sigma}_{12}$ in terms of λ_{12} and $\bar{\lambda}_{12}$, we readily find that the above relations imply that

$$\begin{aligned} \Sigma_{12}^i &= \Sigma_2^N \partial_N \, \Sigma_1^i \, - \frac{1}{2} \, \Sigma_2^N \, \tilde{\partial}^i \, \Sigma_{1N} \, - \, (1 \leftrightarrow 2) \, , \\ \tilde{\Sigma}_{12i} &= \Sigma_2^N \partial_N \, \tilde{\Sigma}_{1i} - \frac{1}{2} \, \Sigma_2^N \, \partial_i \, \Sigma_{1N} \, - \, (1 \leftrightarrow 2) \, . \end{aligned} \tag{3.14}$$

These two relations are summarized by

$$\Sigma_{12}^M = \Sigma_{[2}^N \partial_N \, \Sigma_{1]}^M - \frac{1}{2} \, \Sigma_{[2}^N \, \partial^M \, \Sigma_{1]N} \, . \tag{3.15}$$

The algebra is background independent: the background E_{ij} has dropped out through the use of appropriate gauge parameters. The algebra (3.15) coincides with the one discussed by Siegel in [4].

Equation (3.15) defines the C-bracket (1.8) via $\Sigma_{12} = -[\Sigma_1, \Sigma_2]_C$. For fields A^M, B^M on $\hat{M}$, we have

$$[A, B]_C = [A, B] - \frac{1}{2} A^P (\partial' B_P) + \frac{1}{2} (\partial' A^P) B_P \, . \tag{3.16}$$

We have introduced the notation ∂' for the derivative $\partial^M = \eta^{MN} \partial_N$ with raised index. Here $[A, B]$ is the familiar Lie bracket on the doubled space $\hat{M}$. If the Lie bracket were the only term on the right hand side we would have the algebra of diffeomorphisms on $\hat{M}$. This is not the case, due to the extra terms depending on the metric η which lead to new features that will be explored in later sections.

The gauge transformations that leave the fields invariant have parameters

$$\Sigma^M = \eta^{MN} \partial_N \chi \, . \tag{3.17}$$

As discussed in the introduction, the computation of the gauge algebra on fields only determines the algebra up to such terms, so that Σ_{12}^M is only determined up to the addition of a term $\eta^{MN} \partial_N \chi_{12}$. If this term is to be constructed from Σ_1 and Σ_2 and involve no further derivatives, χ_{12} must be of the form

$$\chi_{12} = \Omega_{PQ} \, \Sigma_{[1}^P \, \Sigma_{2]}^Q \, , \tag{3.18}$$

for some matrix $\Omega_{PQ} = -\Omega_{QP}$. The general gauge algebra is then

$$\Sigma_{12}^M = \Sigma_{[2}^N \partial_N \, \Sigma_{1]}^M - \frac{1}{2} \, \eta^{MN} \eta_{PQ} \, \Sigma_{[2}^P \, \partial_N \, \Sigma_{1]}^Q \, + \eta^{MN} \partial_N \left(\Omega_{PQ} \Sigma_{[1}^P \Sigma_{2]}^Q \right) \, . \tag{3.19}$$

In principle Ω_{PQ} could depend on x and $\tilde{x}$, but if the algebra is to have structure constants, as opposed to field-dependent structure functions, it should be a constant matrix. Any non-zero choice of Ω_{PQ} will not be invariant under $O(D, D)$, so $O(D, D)$ covariance requires setting $\Omega_{PQ} = 0$.

4 Dilaton, scalars, and vectors

The gauge transformation (1.3) of the dilaton can be written covariantly as

$$\delta d = -\frac{1}{2}\partial_N \Sigma^N + \Sigma^N \partial_N \, d \,. \tag{4.1}$$

A short calculation then shows that, for restricted fields, these transformations satisfy the algebra

$$\big[\delta_{\Sigma_1}\,, \delta_{\Sigma_2}\big] d = -\frac{1}{2}\,\partial_N \Sigma_{12}^N + \Sigma_{12}^N\,\partial_N \, d\,, \tag{4.2}$$

with Σ_{12} defined in (3.15), so that this is precisely the same gauge algebra found for the transformations of e_{ij}. For e_{ij} we needed to add extra terms to the algebra in order to close the algebra, but for d the transformations (4.1) close to give precisely the algebra (3.15). We shall take the transformations of d to be (4.1) without any higher order corrections.

The transformation (4.1) can be written as

$$\delta e^{-2d} = \partial_N \big(\Sigma^N \, e^{-2d}\big)\,. \tag{4.3}$$

which is the same as the transformation of a density $\exp(-2d)$ under a diffeomorphism of $\hat{M}$ with infinitesimal parameter Σ. These transformations, of course, satisfy the algebra of diffeomorphisms

$$\big[\delta_{\Sigma_1}\,, \delta_{\Sigma_2}\big] = -\delta_{\check{\Sigma}_{12}} \tag{4.4}$$

where

$$\check{\Sigma}_{12}^M \, \equiv \, \Sigma_{[2}^N \, \partial_N \, \Sigma_{1]}^M \tag{4.5}$$

is the Lie bracket $[\Sigma_2, \Sigma_1]$. In fact the transformation of d is consistent with both the diffeomorphism algebra and the C-algebra (3.15) because for restricted fields use of (3.10) leads to

$$\partial_M \Sigma_{12}^M = \partial_M \check{\Sigma}_{12}^M \quad \text{and} \quad \Sigma_{12}^M \partial_M = \check{\Sigma}_{12}^M \partial_M \,. \tag{4.6}$$

Then the algebra of transformations on d can also be written as

$$\big[\delta_{\Sigma_1}\,, \delta_{\Sigma_2}\big] d = -\frac{1}{2}\,\partial_N \, \check{\Sigma}_{12}^N + \check{\Sigma}_{12}^N \, \partial_N \, d\,. \tag{4.7}$$

It is clear from the above discussion that one can define scalars as well as densities in this theory. A scalar R is required to transform as

$$\delta_\Sigma R = \Sigma^N \partial_N \, R\,. \tag{4.8}$$

The resulting gauge algebra (4.4) is that of diffeomorphisms, and this is consistent with (3.15) since $\check{\Sigma}_{12}^N \partial_N R = \Sigma_{12}^N \partial_N R$ for restricted fields.

With a scalar $R(e, d)$ built using the fields e_{ij} and d, a gauge invariant action (for restricted fields) could be constructed as

$$S = \int dx d\tilde{x} \, e^{-2d} \, R(e, d)\,. \tag{4.9}$$

Since the fields are restricted to some D-dimensional null torus N, the integral in the action could be restricted to N. To leading order in the fields, a suitable candidate with two derivatives is

$$R(e,d) = 4D^2 d + D^i \bar{D}^j e_{ij} + \text{quadratic in fields}\,. \tag{4.10}$$

An all orders construction of this scalar would be very useful. In the work [4] of Siegel, a scalar is constructed and presumably should agree with $R(e,d)$, once suitable gauge conditions are imposed to eliminate the extra gauge degrees of freedom in that formulation.

We conclude with a brief discussion of vectors. A vector field V^M on $\hat{M}$ transforms under an infinitesimal diffeomorphism on $\hat{M}$ with the Lie bracket $\delta_\Sigma V = [\Sigma, V]$. The parameter Σ^M is also a vector field on $\hat{M}$ and the Jacobi identities ensure this is a representation of the diffeomorphism algebra. It is straightforward to see that such a vector

$$V^M \equiv \begin{pmatrix} \tilde{v}_i(x,\tilde{x}) \\ v^i(x,\tilde{x}) \end{pmatrix}, \tag{4.11}$$

restricted to M results in a $v^i(x)$ that is a vector field on M but gives a $\tilde{v}_i(x)$ that does not transform as a 1-form, but rather as a scalar under diffeomorphisms of M. This means that despite the suggestive notation, our gauge parameters Σ^M should not be thought of as conventional vector fields on $\hat{M}$, as their restriction gives a vector field $\xi(x)$ and a 1-form $\tilde{\xi}(x)$, not a vector and scalar. In fact, the reducibility of the symmetry means that, for the restriction to M, $\tilde{\xi}$ and $\tilde{\xi}+d\alpha$ generate the same transformations. Then $\tilde{\xi}$ and $\tilde{\xi}+d\alpha$ can be regarded as equivalent, so that $\tilde{\xi}$ is more properly thought of as a 1-form connection on M.

It is natural to attempt a generalisation of vectors using the C-bracket. A C-vector V on $\hat{M}$ would then transform as

$$\delta_\Sigma V = [\Sigma\,, V]_{{}_C}\,. \tag{4.12}$$

The algebra of these transformations would be exactly (3.15) if the Jacobi identity held for the C-bracket. Since it does not, and the Jacobiator is of the form $\partial^M\chi$, the consistency of the algebra requires that vectors are only defined up to the equivalence $V^M \sim V^M + \partial^M\chi$, so that V_M are the components of a connection one-form on $\hat{M}$. In particular, the gauge parameters $\Sigma^M(X)$ are such C-vectors on $\hat{M}$, as parameters Σ^M and $\Sigma^M + \partial^M\chi$ define the same transformations and so are equivalent.

5 Restricted fields

The original space $M = T^D$ has coordinates x^i and the dual space $\tilde{M} = T^D$ has coordinates $\tilde{x}_i$. Together they form the doubled space $\hat{M} = M \times \tilde{M} = T^{2D}$. The parameters $\xi(x,\tilde{x})$ and $\tilde{\xi}(x,\tilde{x})$ combine to form $\Sigma^M(X)$. The metric η of signature (D,D) is $ds^2 = 2dx^i d\tilde{x}_i\,$. Fields A restricted to satisfy

$$\frac{\partial}{\partial \tilde{x}_i} A = 0\,, \tag{5.1}$$

will be referred to as geometric, as they are fields on the spacetime M.

We now consider the restriction to a general isotropic subspace, namely, a D-dimensional torus submanifold $N \subset \hat{M}$ that is null with respect to η. Then $\hat{M} = N \times \tilde{N}$, where $\tilde{N}$ is another D-dimensional null torus. We introduce periodic coordinates y^i on N and $\tilde{y}_i$ on $\tilde{N}$ $(i, j = 1, \ldots, D)$ so that the metric takes the form $ds^2 = 2dy^i d\tilde{y}_i$. Following [11, 12], we can describe the choice of N with constant projectors Π and $\tilde{\Pi}$:

$$y^i = \Pi^i{}_M X^M, \qquad \tilde{y}_i = \tilde{\Pi}_{iM} X^M . \tag{5.2}$$

As these preserve the metric and respect the periodicities of all coordinates, they define an $O(D, D; \mathbb{Z})$ transformation Φ:

$$\begin{pmatrix} \tilde{y} \\ y \end{pmatrix} = \Phi \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}, \qquad \Phi^I{}_J = \begin{pmatrix} \tilde{\Pi}_{iJ} \\ \Pi^i{}_J \end{pmatrix} . \tag{5.3}$$

The restriction of fields now takes the form

$$\frac{\partial}{\partial \tilde{y}_i} A = 0 . \tag{5.4}$$

Restricted parameters $\Sigma^M(y)$ are fields on N and the projectors can be used to decompose $\Sigma^M(y)$ into a vector field $\zeta^i(y)$ on N and a one-form field $\tilde{\zeta}_i(y)$ on N:

$$\zeta^i = \Pi^i{}_M \Sigma^M, \qquad \tilde{\zeta}_i = \tilde{\Pi}_{iM} \Sigma^M . \tag{5.5}$$

The restriction (5.4) to a subspace N clearly breaks the $O(D, D; \mathbb{Z})$ symmetry. For the restriction to the spacetime M, the constraint $\frac{\partial}{\partial \tilde{x}_i} A = 0$ (see (5.1)) is preserved by the $GL(D; \mathbb{Z})$ subgroup of $O(D, D; \mathbb{Z})$

$$g = \begin{pmatrix} a & 0 \\ 0 & \tilde{a} \end{pmatrix} , \tag{5.6}$$

where $a \in GL(D; \mathbb{Z})$ and $\tilde{a} \equiv (a^t)^{-1}$. Indeed, these transformations simply rotate the ∂_i and $\tilde{\partial}^i$ derivatives among themselves. The constraint (5.1) is also preserved by the B-transformations

$$g = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} . \tag{5.7}$$

where θ is a constant integer-valued antisymmetric matrix. This transformation acts on the derivatives as

$$\partial_i \to \partial_i + \theta_{ij} \, \tilde{\partial}^j, \qquad \tilde{\partial}^i \to \tilde{\partial}^i, \tag{5.8}$$

making it clear that the constraint is unchanged.

The above $GL(D; \mathbb{Z})$ and B-transformations form the 'geometric' subgroup Γ of $O(D, D; \mathbb{Z})$ [11]. The restriction (5.4) to N is preserved by a group Γ_N conjugate to the geometric subgroup Γ:

$$\Gamma_N = \{ \Phi g \Phi^{-1} : g \in \Gamma \} . \tag{5.9}$$

For $M = \mathcal{M}_n \times T^d$, the product of n-dimensional Minkowski space $\mathcal{M}_n$ with coordinates x^μ and a d-torus with coordinates x^a, we introduce a dual space $\tilde{M} = \tilde{\mathcal{M}}_n \times T^d$ with a dual n-dimensional Minkowski space $\tilde{\mathcal{M}}_n$ with coordinates $\tilde{x}_\mu$ and a dual d-torus with coordinates $\tilde{x}_a$. The doubled space is $\hat{M} = M \times \tilde{M} = \mathcal{M}_n \times \tilde{\mathcal{M}}_n \times T^{2d}$. Absence of winding in the non-compact directions requires that all fields and parameters are independent of $\tilde{x}_\mu$. Restricted fields must then be fields on a space $N = \mathcal{M}_n \times N_d \subset \hat{M}$, where N_d is a d-dimensional null torus subspace of the double torus T^{2d}. Then there is a null torus $\tilde{N}_d$ so that $T^{2d} = N_d \times \tilde{N}_d$. If the coordinates of N_d are y^a and those of $\tilde{N}_d$ are $\tilde{y}_a$, then fields restricted to N are independent of $\tilde{y}_i = (\tilde{x}_\mu, \tilde{y}_a)$.

6 Reducibility and the Jacobiator

In this section, we discuss further the issues concerning a gauge algebra with a non-vanishing Jacobiator and the ambiguities in the gauge algebra. Consider then a (possibly infinite dimensional) closed algebra

$$[T_A, T_B] = f_{AB}{}^C T_C \,, \tag{6.1}$$

with constant $f_{AB}{}^C$. If the T_A are to be realised as a set of classical infinitesimal symmetry transformations on a set of fields, with $[T_A, T_B]$ the commutator of two such transformations, then it is necessary that

$$[[T_A, T_B], T_C] + \text{cyclic permutations} \tag{6.2}$$

vanishes when acting on fields. This will be the case if the structure constants satisfy the Jacobi identity so that we have a Lie algebra. However, suppose that (6.2) is not zero, but takes values in a closed sub-algebra with basis Z_α. Let a basis for the remaining generators be t_a, so that $T_A = \{t_a, Z_\alpha\}$. Then the structure constants satisfy

$$f_{[AB}{}^D f_{C]D}{}^a = 0, \qquad f_{[AB}{}^D f_{C]D}{}^\alpha = -2 g_{ABC}{}^\alpha \,, \tag{6.3}$$

for some constants $g_{ABC}{}^\alpha$, so that the generators satisfy

$$[[T_A, T_B], T_C] + \text{cyclic permutations} = g_{ABC}{}^\alpha Z_\alpha \,. \tag{6.4}$$

Then the algebra can be realised as classical infinitesimal symmetry transformations on a set of fields provided that the subalgebra of the Z's leaves all the fields invariant, so the symmetry is reducible. If each Z_α gives zero when acting on every field, then (6.2) will give zero acting on fields, as required.[4]

A general infinitesimal transformation will be a linear combination $\Sigma^A T_A$ of the T_A for some Σ^A. As usual, the algebra (6.1) defines a bracket of the parameters Σ^A

$$[\Sigma_1\,, \Sigma_2]^D = f_{AB}{}^D \Sigma_1^A \Sigma_2^B \,, \tag{6.5}$$

with Jacobiator

$$J(\Sigma_1, \Sigma_2, \Sigma_3)^E = \frac{1}{2} \Sigma_1^A \Sigma_2^B \Sigma_3^C \, f_{[AB}{}^D f_{C]D}{}^E \,, \tag{6.6}$$

[4] In the Batalin-Vilkovisky master action, $g_{ABC}{}^\alpha$ appears in the quartic coupling of three ghost fields to the anti-field of a 2nd generation ghost.

so that

$$J(\Sigma_1, \Sigma_2, \Sigma_3)^a = 0\,, \qquad J(\Sigma_1, \Sigma_2, \Sigma_3)^\alpha = -\Sigma_1^A \Sigma_2^B \Sigma_3^C g_{ABC}{}^\alpha\,. \tag{6.7}$$

Our symmetry algebra is of precisely this type. The parameters are $\Sigma^M(x,\tilde{x})$ so that the corresponding generators are $T_M(x,\tilde{x})$, with the classical transformation δ_Σ with infinitesimal parameter $\Sigma^M(x,\tilde{x})$ written as $\int dxd\tilde{x}\, \Sigma^M(x,\tilde{x})T_M(x,\tilde{x})$. It is convenient to write this as $\Sigma^A T_A$ with A a composite index representing the discrete index M and the continuous variables $x, \tilde{x}$ and summation over A representing summation over M and integration over $x, \tilde{x}$. The structure constants $f_{AB}{}^C$ are then defined by the C-bracket through (6.5) where $[\Sigma_1\,, \Sigma_2]$ is minus the C-bracket. There is a subalgebra of generators Z_α generating transformations that leave the fields invariant, consisting of transformations $\Sigma^A T_A$ with Σ of the form (1.9). The structure constants for the Courant bracket and the C-bracket satisfy relations of the form (6.3) so that the Jacobi identitites are violated by terms in the Z-algebra. As these do not act on the fields, the algebra can be consistently realised on fields. It is tempting to try to set the Z-generators to zero in some way, but there does not appear to be a local covariant way of doing this. Reducibility is intimately related to the failure of the Jacobi identities.

The algebra (6.1) can be written as

$$[T_A, T_B] = f_{AB}{}^c t_c + f_{AB}{}^\alpha Z_\alpha\,. \tag{6.8}$$

The calculation of the algebra of gauge transformations $[\delta_{\Sigma_1}, \delta_{\Sigma_2}]$ acting on fields only determines the structure constants $f_{AB}{}^c$ but leaves the $f_{AB}{}^\alpha$ completely undetermined as Z_α does not act on fields. In our case, demanding $O(D,D)$ covariance fixes the $f_{AB}{}^\alpha$ completely and gives the algebra of the C-bracket. We shall see, however, that other non-covariant choices are possible, including one which gives an algebra that does satisfy the Jacobi identity.

7 The Courant bracket

The Courant bracket is defined on smooth sections of $T \oplus T^*$, where T is the tangent bundle of a manifold and T^* the cotangent bundle. Such a section is the formal sum $A + \alpha$ of a vector field A and a 1-form field α. For two such sections $A + \alpha$ and $B + \beta$, the Courant bracket is the skew-symmetric bracket given by

$$[A + \alpha, B + \beta]_{\rm Cour} = [A, B] + \mathcal{L}_A \beta - \mathcal{L}_B \alpha - \frac{1}{2} d(i_A \beta - i_B \alpha). \tag{7.1}$$

Note that for two such sections, there is also a natural inner product

$$\langle A + \alpha, B + \beta \rangle = i_A \beta + i_B \alpha\,, \tag{7.2}$$

which is of signature (D, D) on a space of dimension D. This is the flat metric η.

The Courant bracket is not a Lie bracket since it fails to satisfy the Jacobi identity. The Jacobiator (see (1.11)) of sections P, Q, R of $T \oplus T^*$ is given by [8, 9]:

$$J(P, Q, R) = d\, N(P, Q, R), \tag{7.3}$$

where N is the 'Nijenhuis operator' given by

$$N(P,Q,R)=\frac{1}{6}\Big(\langle[P,Q]_{\rm Cour},R\rangle+\langle[Q,R]_{\rm Cour}\,,P\rangle+\langle[R,P]_{\rm Cour},Q\rangle\Big)\,. \tag{7.4}$$

While the gauge algebra for diffeomorphisms is given by the Lie bracket, that for diffeomorphisms plus b-field gauge transformations has a subtlety that is suggestive of the Courant bracket. Under an infinitesimal diffeomorphism with a vector field parameter ξ and a B-field gauge transformation with a one-form parameter $\tilde{\xi}$ we have

$$\delta g=\mathcal{L}_\xi g\,,\quad \text{and}\quad \delta b=\mathcal{L}_\xi b+d\tilde{\xi}\,, \tag{7.5}$$

for a metric g and a two-form field b. The symmetry is reducible: replacing $\tilde{\xi}\to\tilde{\xi}+d\sigma$ leaves the transformations unchanged and constitutes a "symmetry for a symmetry". The computation of the algebra quickly gives the first three terms of the right-hand side of (7.1). The last term, $d(\ldots)$, can be added with *arbitrary* coefficient, as it represents an ambiguity: the gauge transformations are unchanged when the B-field gauge parameter changes by an exact term. This is the ambiguity discussed in the introduction and section 6. In (6.8) the structure constants $f_{AB}{}^\alpha$ are completely undetermined as Z_α does not act on the fields. Here Z_α generate transformations with $\tilde{\xi}=d\sigma$. Hence the gauge algebra acting on fields gives the bracket

$$[A+\alpha,B+\beta]_\kappa=[A,B]+\mathcal{L}_A\beta-\mathcal{L}_B\alpha-\frac{1}{2}\,\kappa\,d(i_A\beta-i_B\alpha). \tag{7.6}$$

with arbitrary coefficient κ. Taking $\kappa=0$ gives an algebra that satisfies the Jacobi identity. Any $\kappa\neq 0$ gives a non-zero Jacobiator equal to an exact one-form, so that the resulting gauge parameter does not act on any fields. Taking $\kappa=1$ gives the Courant bracket.

We can now show that the C-bracket reduces to the Courant bracket when the parameters are required to be independent of $\tilde{x}$. To do this we write out the terms in (1.8)

$$\big([\Sigma_1,\Sigma_2]_{_C}\big)^M=\Sigma_1^N\partial_N\Sigma_2^M-\Sigma_2^N\partial_N\Sigma_1^M\;-\frac{1}{2}\,\Sigma_1^N\,\partial^M\Sigma_{2N}+\frac{1}{2}\,\Sigma_2^N\,\partial^M\Sigma_{1N}\,. \tag{7.7}$$

To find an explicit formula in terms of $\xi,\tilde{\xi}$ and $\partial,\tilde{\partial}$ we use the definitions at the beginning of section 3. Since the parameters are restricted to be independent of $\tilde{x}$, all terms involving $\tilde{\partial}^i$ in (7.7) vanish and $\xi^i(x)$ is a vector field on M and $\tilde{\xi}_i(x)$ is a 1-form field. Then

$$\big([\Sigma_1,\Sigma_2]_{_C}\big)^M=\begin{pmatrix}([\Sigma_1,\Sigma_2]_{_C})_i\\ ([\Sigma_1,\Sigma_2]_{_C})^i\end{pmatrix}\,, \tag{7.8}$$

where

$$\big([\Sigma_1,\Sigma_2]_{_C}\big)^i=\xi_1^j\partial_j\xi_2^i-\xi_2^j\partial_j\xi_1^i=(\mathcal{L}_{\xi_1}\xi_2)^i=\big([\xi_1,\xi_2]\big)^i\,, \tag{7.9}$$

is the Lie bracket of two vector fields while

$$\begin{aligned}\big([\Sigma_1,\Sigma_2]_{_C}\big)_i&=\xi_1^j\partial_j\tilde{\xi}_{2i}\;-\frac{1}{2}(\xi_1^j\partial_i\tilde{\xi}_{2j}-\tilde{\xi}_{2j}\partial_i\xi_1^j)\quad-(1\leftrightarrow 2)\\ &=\xi_1^j\partial_j\tilde{\xi}_{2i}\;+(\partial_i\xi_1^j)\tilde{\xi}_{2j}\;-\frac{1}{2}\partial_i(\xi_1^j\,\tilde{\xi}_{2j})\quad-(1\leftrightarrow 2)\\ &=\Big(\;\mathcal{L}_{\xi_1}\tilde{\xi}_2-\frac{1}{2}\,d(i_{\xi_1}\tilde{\xi}_2)\;\Big)_i\quad-(1\leftrightarrow 2)\,.\end{aligned} \tag{7.10}$$

Rewriting the above results in terms of the formal sum $\xi + \tilde{\xi}$, we find that the C-bracket for parameters independent of $\tilde{x}$ is precisely the Courant bracket $[\xi_1 + \tilde{\xi}_1, \xi_2 + \tilde{\xi}_2]_{\rm Cour}$. A very similar calculation shows that for parameters restricted to N by (5.4), the C-bracket $[\Sigma_1, \Sigma_2]_C$ is precisely the Courant bracket $[\zeta_1 + \tilde{\zeta}_1, \zeta_2 + \tilde{\zeta}_2]_{\rm Cour}$ on N where the parameters have been decomposed into vectors ζ and one-forms $\tilde{\zeta}$ on N as in (5.5). This is of course as was to be expected, as the restriction to N is obtained from the restriction to M by the $O(D, D; \mathbb{Z})$ transformation (5.3) which is a symmetry of the C-bracket, so that the M restriction and the N restriction are isomorphic.

As discussed at the end of section 3, calculating the gauge algebra on the fields gives the gauge algebra (3.19) with any choice of 2-form Ω_{PQ}. If we choose

$$\Omega = \gamma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{7.11}$$

then we obtain

$$[\Sigma_1, \Sigma_2]_{C\,i} = \Big(\mathcal{L}_{\xi_1} \tilde{\xi}_2 - \frac{1}{2} \kappa \, d(i_{\xi_1} \tilde{\xi}_2) \Big)_i - (1 \leftrightarrow 2) \, , \quad \text{with} \quad \kappa = 1 + 4\gamma \, . \tag{7.12}$$

The Courant bracket has been replaced by the κ-bracket (7.6). This is of course to be expected, as the two systems are related by field redefinitions, as we showed in section 2.

These brackets also arise from current algebras. In the canonical treatment of the string in flat space with coordinates x^i, the canonical variables are loops $x^i(\sigma)$ with conjugate momenta $p_i(\sigma)$. Here σ is a periodic coordinate on the string. The currents

$$J_{\xi + \tilde{\xi}} = \xi^i p_i + \tilde{\xi}_i \frac{dx^i}{d\sigma} \, , \tag{7.13}$$

satisfy a canonical current algebra

$$[J_{\xi+\tilde{\xi}}, J_{\zeta+\tilde{\zeta}}] = J_{\chi+\tilde{\chi}} + \dots \tag{7.14}$$

where

$$\chi + \tilde{\chi} = [\xi + \tilde{\xi}, \zeta + \tilde{\zeta}] \tag{7.15}$$

defines a bracket. In [13] this calculation was done for the case in which ξ and $\tilde{\xi}$ depend on $x(\sigma)$ and it was found that the resulting bracket could be the Courant bracket. In fact there is an ambiguity in the calculation and the general result is in fact the κ-bracket given above. Siegel had done essentially the same calculation earlier [4] in the more general context in which ξ and $\tilde{\xi}$ depend on both x and $\tilde{x}$ but are restricted to take values on some N. In this case, he used a duality covariant formalism and found precisely the C-bracket.

8 The C-bracket

We have seen that the C-bracket of fields restricted to the null D-dimensional torus N in $\hat{M} = M \times \tilde{M}$ is in fact the Courant bracket on $(T \oplus T^*)N$. In this section we write the C-bracket in terms of derivatives with respect to coordinates x^i of M and coordinates

$\tilde{x}_i$ of $\tilde{M}$. We decompose each Σ^M on $\hat{M}$ into $\xi^i(x,\tilde{x})$ and $\tilde{\xi}_i(x,\tilde{x})$ as in (3.6). We find a symmetric structure in which ξ and $\tilde{\xi}$ are treated similarly. If Σ is restricted to M, then $\xi^i(x)$ and $\tilde{\xi}_i(x)$ are vectors and one-forms on M.

The asymmetry in the treatment of vectors and one-forms in the Courant bracket led the authors of [9] to introduce a Courant-like bracket treating them symmetrically. With sections A, B of a bundle L and sections α, β of a dual bundle L^*, where (L, L^*) form a so-called Lie bi-algebroid, the bracket takes the form:

$$\begin{aligned}[A+\alpha, B+\beta] = [A,B] + \mathcal{L}_\alpha B - \mathcal{L}_\beta A + \frac{1}{2}\tilde{d}(i_A\beta - i_B\alpha) \\ + [\alpha,\, \beta] + \mathcal{L}_A\beta - \mathcal{L}_B\alpha - \frac{1}{2}d(i_A\beta - i_B\alpha).\end{aligned} \tag{8.1}$$

This bracket is the key element in making $L\oplus L^*$ into a "Courant algebroid". We show that the C-bracket, with natural definitions associated with the space $M\times\tilde{M}$, takes precisely the form (8.1). This is true even if the fields are not restricted! However, it is only when the fields are restricted to N that this becomes an example of the general setup of [9], giving a Courant algebroid over N.[5] We conclude with a computation of the Jacobiator of the C-bracket. We assume restricted fields, but make no explicit reference to the choice of N. We show that the Jacobiator is a trivial gauge parameter, as required from the discussion given in the introduction and section 6.

We begin by considering the C-bracket, which takes the form (7.7):

$$\begin{aligned}\big([\Sigma_1,\Sigma_2]_{_C}\big)^M &= \big([\Sigma_1,\Sigma_2]\big)^M \; - \frac{1}{2}\,\Sigma^N_{[1}\,\partial^M\Sigma_{2]N} \\ &= \Sigma_1^N\partial_N\Sigma_2^M - \Sigma_2^N\partial_N\Sigma_1^M \; - \frac{1}{2}\,\Sigma_1^N\,\partial^M\Sigma_{2N} + \frac{1}{2}\,\Sigma_2^N\,\partial^M\Sigma_{1N}\,,\end{aligned} \tag{8.2}$$

where $[\Sigma_1,\Sigma_2]$ is the Lie bracket for the doubled space $\hat{M}$. Each Σ is decomposed into a ξ^i and a $\tilde{\xi}_i$ as in (3.6). For notational convenience we write $\Sigma = \xi + \tilde{\xi}$ formally adding together ξ and $\tilde{\xi}$. The bracket can then be evaluated as

$$[\xi_1 + \tilde{\xi}_1\,, \xi_2 + \tilde{\xi}_2]_{_C} = [\xi_1, \xi_2]_{_C} + [\xi_1, \tilde{\xi}_2]_{_C} + [\tilde{\xi}_1\,, \xi_2]_{_C} + [\tilde{\xi}_1, \tilde{\xi}_2]_{_C}\,. \tag{8.3}$$

For $[\xi_1,\xi_2]_{_C}$ and $[\tilde{\xi}_1,\tilde{\xi}_2]_{_C}$, the second term in the algebra (8.2) vanishes because this term necessarily couples a ξ to a $\tilde{\xi}$ as the metric η is off-diagonal. Therefore, for both these computations the C-bracket reduces to the Lie bracket on the doubled space. Let us first consider $[\xi_1,\xi_2]_{_C}$. We have

$$\big([\xi_1,\xi_2]_{_C}\big)^M = \big([\xi_1,\xi_2]\big)^M = \begin{pmatrix}([\xi_1,\xi_2]_{_C})_i \\ ([\xi_1,\xi_2]_{_C})^i\end{pmatrix} = \xi_1^j\partial_j\begin{pmatrix}0\\ \xi_2^i\end{pmatrix} - \xi_2^j\partial_j\begin{pmatrix}0\\ \xi_1^i\end{pmatrix}\,. \tag{8.4}$$

We thus conclude that

$$\begin{aligned}\big([\xi_1,\xi_2]_{_C}\big)_i &= \;\; 0\,, \\ \big([\,\xi_1,\xi_2\,]_{_C}\big)^i &= \xi_1^j\partial_j\xi_2^i - \xi_2^j\partial_j\xi_1^i\,.\end{aligned} \tag{8.5}$$

[5] (TN, T^*N) form a Lie bi-algebroid over N, and their sum is a Courant algebroid $(T\oplus T^*)N$ over N with bracket (8.1), which is the Courant bracket on $TN\oplus T^*N$.

With the Lie derivative

$$(\mathcal{L}_{\xi_1}\xi_2)^i = \xi_1^j\partial_j\xi_2^i - \xi_2^j\partial_j\xi_1^i\,, \tag{8.6}$$

and the bracket $([\xi_1,\xi_2])^i = (\mathcal{L}_{\xi_1}\xi_2)^i$, we have

$$[\xi_1,\xi_2]_{{}_C} = \mathcal{L}_{\xi_1}\xi_2 = [\xi_1,\xi_2]\,. \tag{8.7}$$

For fixed $\tilde{x}$, this would be the usual Lie derivative and Lie bracket on M.

Let us now consider $[\tilde\xi_1,\tilde\xi_2]_{{}_C}$. This time we get

$$\left([\tilde\xi_1,\tilde\xi_2]_{{}_C}\right)^M = \left([\tilde\xi_1,\tilde\xi_2]\right)^M = \begin{pmatrix}([\tilde\xi_1,\tilde\xi_2]_{{}_C})_i\\ ([\tilde\xi_1,\tilde\xi_2]_{{}_C})^i\end{pmatrix} = \tilde\xi_{1j}\tilde\partial^j\begin{pmatrix}\tilde\xi_{2i}\\ 0\end{pmatrix} - \tilde\xi_{2j}\tilde\partial^j\begin{pmatrix}\tilde\xi_{1i}\\ 0\end{pmatrix}. \tag{8.8}$$

giving

$$\begin{aligned}\left([\tilde\xi_1,\tilde\xi_2]_{{}_C}\right)_i &= \tilde\xi_{1j}\tilde\partial^j\tilde\xi_{2i} - \tilde\xi_{2j}\tilde\partial^j\tilde\xi_{1i}\,,\\ \left([\tilde\xi_1,\tilde\xi_2]_{{}_C}\right)^i &= \;0\,.\end{aligned} \tag{8.9}$$

For any α_i and β_i we define the Lie derivative along α of β as

$$(\mathcal{L}_\alpha\beta)_i \equiv \alpha_j\tilde\partial^j\,\beta_i - (\tilde\partial^j\alpha_i)\beta_j = \big([\alpha\,,\beta]\big)_i\,. \tag{8.10}$$

The Lie-bracket $[\alpha\,,\beta]$ has a lower index and is computed using only $\tilde{x}$-derivatives. Then we have

$$[\tilde\xi_1,\tilde\xi_2]_{{}_C} = \mathcal{L}_{\tilde\xi_1}\tilde\xi_2 = [\tilde\xi_1,\tilde\xi_2]\,. \tag{8.11}$$

The mixed terms bring new features. Let us compute $[\xi_1,\tilde\xi_2]_{{}_C}$. This time we get contributions from all terms in the C-bracket:

$$\left([\xi_1,\tilde\xi_2]_{{}_C}\right)^M = \begin{pmatrix}([\xi_1,\tilde\xi_2]_{{}_C})_i\\ ([\xi_1,\tilde\xi_2]_{{}_C})^i\end{pmatrix} = \xi_1^j\partial_j\begin{pmatrix}\tilde\xi_{2i}\\ 0\end{pmatrix} - \tilde\xi_{2j}\tilde\partial^j\begin{pmatrix}0\\ \xi_1^i\end{pmatrix} - \frac{1}{2}\,\xi_1^j\begin{pmatrix}\partial_i\\ \tilde\partial^i\end{pmatrix}\tilde\xi_{2j} + \frac{1}{2}\,\tilde\xi_{2j}\begin{pmatrix}\partial_i\\ \tilde\partial^i\end{pmatrix}\xi_1^j\,. \tag{8.12}$$

We thus get

$$\begin{aligned}\left([\xi_1,\tilde\xi_2]_{{}_C}\right)_i &= \;\xi_1^j\partial_j\tilde\xi_{2i} - \frac{1}{2}(\xi_1^j\partial_i\tilde\xi_{2j} - \tilde\xi_{2j}\partial_i\xi_1^j) \;=\; \xi_1^j\partial_j\tilde\xi_{2i} + (\partial_i\xi_1^j)\tilde\xi_{2j} - \frac{1}{2}\partial_i(\xi_1^j\,\tilde\xi_{2j})\,,\\ \left([\,\xi_1,\tilde\xi_2\,]_{{}_C}\right)^i &= -\tilde\xi_{2j}\tilde\partial^j\xi_1^i - \frac{1}{2}(\xi_1^j\tilde\partial^i\tilde\xi_{2j} - \tilde\xi_{2j}\tilde\partial^i\xi_1^j) = -\tilde\xi_{2j}\tilde\partial^j\xi_1^i - (\tilde\partial^i\tilde\xi_{2j})\xi_1^j + \frac{1}{2}\tilde\partial^i(\xi_1^j\,\tilde\xi_{2j})\,.\end{aligned} \tag{8.13}$$

A few natural definitions help rewrite this more clearly. Given A^i,α_i we define

$$\begin{aligned}(\mathcal{L}_A\alpha)_i &\equiv A^j\partial_j\,\alpha_i + (\partial_iA^j)\alpha_j\,,\\ (\mathcal{L}_\alpha A)^i &\equiv \alpha_j\tilde\partial^j A^i + (\tilde\partial^i\alpha_j)A^j\,.\end{aligned} \tag{8.14}$$

We also define exterior derivatives d and $\tilde{d}$. Acting on a function S they give

$$\begin{aligned}(dS)_i &= \partial_iS\,, \quad (dS)^i = 0\,,\\ (\tilde{d}S)^i &= \tilde\partial^iS\,, \quad (\tilde{d}S)_i = 0\,.\end{aligned} \tag{8.15}$$

Finally, we define contractions and dual contractions,

$$\alpha_i A^i = i_A\alpha = \tilde{\imath}_\alpha A\,. \tag{8.16}$$

One can verify that on any A^i (as in the second equation in (8.14)), we have

$$\tilde{\mathcal{L}}_\alpha = \tilde{d}\,\tilde{\imath}_\alpha + \tilde{\imath}_\alpha\tilde{d}\,. \tag{8.17}$$

This is completely analogous to the formula that gives the action of standard Lie derivatives on forms. We can now return to (8.13) and write

$$\begin{aligned}
\big([\xi_1,\tilde{\xi}_2]_C\big)_i &= \Big(\,\mathcal{L}_{\xi_1}\tilde{\xi}_2 - \frac{1}{2}\,d(i_{\xi_1}\tilde{\xi}_2)\,\Big)_i\,,\\
\big([\,\xi_1,\tilde{\xi}_2\,]_C\big)^i &= \Big(-\mathcal{L}_{\tilde{\xi}_2}\xi_1 + \frac{1}{2}\tilde{d}(\tilde{\imath}_{\tilde{\xi}_2}\xi_1)\Big)^i\,.
\end{aligned} \tag{8.18}$$

The two expressions above are summarized by

$$[\xi_1,\tilde{\xi}_2]_C = -\mathcal{L}_{\tilde{\xi}_2}\xi_1 + \frac{1}{2}\tilde{d}(\tilde{\imath}_{\tilde{\xi}_2}\xi_1) \;+\; \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \frac{1}{2}\,d(i_{\xi_1}\tilde{\xi}_2)\,. \tag{8.19}$$

Using the above result, together with (8.7) and (8.11) we readily evaluate (8.3). The result is

$$\begin{aligned}
[\,\xi_1+\tilde{\xi}_1\,,\,\xi_2+\tilde{\xi}_2\,]_C = [\xi_1,\xi_2] + \mathcal{L}_{\tilde{\xi}_1}\xi_2 - \mathcal{L}_{\tilde{\xi}_2}\xi_1 - \frac{1}{2}\,\tilde{d}\,(\tilde{\imath}_{\tilde{\xi}_1}\xi_2 - \tilde{\imath}_{\tilde{\xi}_2}\xi_1)\\
+\,[\tilde{\xi}_1,\tilde{\xi}_2] + \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1 - \frac{1}{2}\,d\,(i_{\xi_1}\tilde{\xi}_2 - i_{\xi_2}\tilde{\xi}_1).
\end{aligned} \tag{8.20}$$

We see that this is precisely the bracket in (8.1). For gauge parameters that are independent of $\tilde{x}$, so that $\tilde{\partial} = 0$ on all quantities, this reduces to

$$[\xi_1+\tilde{\xi}_1,\xi_2+\tilde{\xi}_2]_C = [\xi_1,\xi_2] + \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1 - \frac{1}{2}d(i_{\xi_1}\tilde{\xi}_2 - i_{\xi_2}\tilde{\xi}_1),$$

which is precisely the Courant bracket (7.1).

Let us now compute the Jacobiator for the C-bracket. For this purpose it is useful to introduce a related product $\circ$ for fields P^M, Q^M on $\hat{M}$, defined by[6]

$$P\circ Q \equiv [P,Q] + (\partial' P^M)Q_M\,. \tag{8.21}$$

It then follows (see (3.16)) that the C-bracket and the $\circ$-product differ by a total derivative:

$$[P,Q]_C = P\circ Q - \frac{1}{2}\,\partial'\,(P^M Q_M)\,. \tag{8.22}$$

(Recall our notation ∂' for the derivative $\partial^M = \eta^{MN}\partial_N$ with raised index.) The inner product (7.2) is $\langle P,Q\rangle \equiv P^P Q_P$ and allows us to write

$$[P,Q]_C = P\circ Q - \frac{1}{2}\,\partial'\langle P,Q\rangle\,. \tag{8.23}$$

[6] For restricted fields, the C-bracket becomes the Courant bracket and the product $\circ$ becomes the Dorfman bracket.

While the $\circ$-product is not skew symmetric, its skew-symmetrisation gives the C-bracket:

$$[P,Q]_C = \frac{1}{2}(P\circ Q - Q\circ P)\,. \tag{8.24}$$

A key property of the $\circ$-product is that it vanishes when the first factor is of the form $\partial^M S$, with S a scalar:

$$\begin{aligned}((\partial' S)\circ P)^M &= [\partial' S, P]^M + (\partial^M\partial^K S)P_K \\ &= (\partial^K S)\partial_K P^M - P^K\partial_K\partial^M S + (\partial^M\partial^K S)P_K \\ &= (\partial^K S)\partial_K P^M = 0\,,\end{aligned} \tag{8.25}$$

where the term in the final line vanishes by the constraint. This property, together with (8.23), gives

$$\big[\,[P,Q]_C\,,R\big]_C = (P\circ Q)\circ R - \frac{1}{2}\partial'\,\langle [P,Q]_C\,,R\rangle\,. \tag{8.26}$$

The $\circ$-product satisfies a Leibnitz identity:

$$P\circ(Q\circ R) = (P\circ Q)\circ R + Q\circ(P\circ R)\,. \tag{8.27}$$

This is verified using the definition (8.21) and the Jacobi identity for the Lie bracket $[\cdot,\cdot]$. After some calculation one is left with a small set of terms, all of which vanish using the constraint that $(\partial^N X)\partial_N Y = 0$ for any X, Y. We now have all the requisite identities. Following the standard steps used to compute the Jacobiator for the Courant bracket (see Proposition 3.16 in [8]), we have

$$\begin{aligned}J(P,Q,R) &= \frac{1}{4}\Big(P\circ(Q\circ R) + \text{c.p.}\Big) \\ &= \frac{1}{4}\Big(\big[\,[P,Q]_C\,,R\big]_C + \frac{1}{2}\partial\,\langle [P,Q]_C\,,R\rangle + \text{c.p.}\Big)\,.\end{aligned} \tag{8.28}$$

Here c.p. stands for cyclic permutation. We thus get

$$J(P,Q,R) = \frac{1}{6}\partial'\Big(\langle [P,Q]_C\,,R\rangle + \text{c.p.}\Big)\,. \tag{8.29}$$

This is of course consistent with the Jacobiator for the Courant bracket given earlier.

9 Conclusions

In this paper we have focused on fields restricted to some null subspace N of the doubled space $\hat{M}$ and investigated the consequences of this restriction for the theory of [1]. Since the space N need not be specified, the field theory has $O(D,D)$ covariance. The results of Siegel [4] also give an $O(D,D)$ covariant field theory defined on such an N. While the theory of [4] involved geometric fields on $\hat{M}$ with additional gauge invariances, ours involves just the fields that arise in closed string field theory. It will be interesting to compare the two approaches further, and in particular the consequences of the failure of the Jacobi identities. The restriction to N implies that these field theories do *not* really include both momentum and winding, as they are T-dual to ones without winding.

For the full gauge algebra of double field theory we are interested in the C-bracket for arbitrary gauge parameters on $\hat{M}$ that satisfy the $\Delta = 0$ constraint but are not necessarily restricted to some null space N. The C-bracket of such fields takes the elegant and symmetric form (8.20), with a projector to $\Delta = 0$ implicit, as in [1]. It would be interesting to investigate the structure of the Jacobiator in this general setup. The results in this paper and the understanding gained on the relation of the C-bracket to the Courant bracket should play an important role in the construction of the full double field theory for fields not restricted to any null subspace.

Our gauge algebra is given by the C-bracket on the doubled space $\hat{M}$. This algebra is distinct from the diffeomorphism algebra on $\hat{M}$ which would be given by the Lie bracket. A striking feature is that, when restricted to *any* N, the gauge parameters Σ^M decompose into a vector field and 1-form on N that are parameters for diffeomorphisms and B-field gauge transformations. As discussed at the end of section 4, our gauge parameters Σ^M are not conventional vector fields on $\hat{M}$ but are instead C-vectors transforming as in (4.12) and identified under the transformations $\Sigma^M \to \Sigma^M + \partial^M \chi$. Although we have diffeomorphism symmetry on every N, the lift to the doubled space $\hat{M}$ gives symmetries with a C-bracket algebra that appear to be distinct from diffeomorphisms on $\hat{M}$ and which will generalise to unrestricted fields. It will be interesting to understand the symmetry structure arising from the C-bracket and the generalisation to unrestricted fields further. We hope to return to these issues in the future.

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A Commutator of gauge transformations

We use matrix notation to rewrite the e_{ij} transformation (2.20) as $\delta_\lambda e = \delta_\lambda^{(0)} e + \delta_\lambda^{(1)} e + \delta_\lambda^{(2)} e$, with

$$\begin{aligned} \delta_\lambda^{(0)} e &= \; M(\lambda, \bar{\lambda})\,, \\ \delta_\lambda^{(1)} e &= \; \mathcal{O}(\lambda, \bar{\lambda})\, e + N(\lambda)\, e \; - e\, \bar{N}(\bar{\lambda})\,, \\ \delta_\lambda^{(2)} e &= -\frac{1}{4}\, e M^t(\lambda, \bar{\lambda})\, e\,. \end{aligned} \tag{A.1}$$

In here we have introduced the differential operator $\mathcal{O}$ and the matrices M, N, and $\bar{N}$ defined by

$$\begin{aligned} \mathcal{O}(\lambda, \bar{\lambda}) &\equiv \frac{1}{2}(\lambda \cdot D + \bar{\lambda} \cdot \bar{D}) \,, \\ M_{ij}(\lambda, \bar{\lambda}) &\equiv D_i \bar{\lambda}_j + \bar{D}_j \lambda_i \,, \\ N_{ij}(\lambda) &\equiv D_i \lambda^j - D^j \lambda_i \,, \\ \bar{N}_{ij}(\bar{\lambda}) &\equiv \bar{D}^i \bar{\lambda}_j - \bar{D}_j \bar{\lambda}^i \,. \end{aligned} \tag{A.2}$$

The gauge algebra requires $\left[\, \delta_{\lambda_1} \,,\, \delta_{\lambda_2} \,\right] = \delta_\Lambda$ and this condition results in

$$\begin{aligned} \delta_\Lambda^{(0)} &= \left[\, \delta_{\lambda_1}^{(0)} \,,\, \delta_{\lambda_2}^{(1)} \,\right] , \\ \delta_\Lambda^{(1)} &= \left[\, \delta_{\lambda_1}^{(1)} \,,\, \delta_{\lambda_2}^{(1)} \,\right] + \left[\, \delta_{\lambda_1}^{(0)} \,,\, \delta_{\lambda_2}^{(2)} \,\right] + \left[\, \delta_{\lambda_1}^{(2)} \,,\, \delta_{\lambda_2}^{(0)} \,\right] , \\ \delta_\Lambda^{(2)} &= \left[\, \delta_{\lambda_1}^{(1)} \,,\, \delta_{\lambda_2}^{(2)} \,\right] + \left[\, \delta_{\lambda_1}^{(2)} \,,\, \delta_{\lambda_2}^{(1)} \,\right] , \\ 0 &= \left[\, \delta_{\lambda_1}^{(2)} \,,\, \delta_{\lambda_2}^{(2)} \,\right] . \end{aligned} \tag{A.3}$$

The first condition requires

$$M(\Lambda, \bar{\Lambda}) \;=\; \mathcal{O}(\lambda_2, \bar{\lambda}_2) M(\lambda_1, \bar{\lambda}_1) + N(\lambda_2) M(\lambda_1) + M(\lambda_2, \bar{\lambda}_2) \bar{N}(\bar{\lambda}_1) \;-\; \{\lambda_1 \leftrightarrow \lambda_2\} \,. \tag{A.4}$$

We have checked this equation works out correctly. The second condition in (A.3), for the algebra to hold with terms linear on the fields, requires the following conditions:

$$\begin{aligned} \mathcal{O}(\Lambda, \bar{\Lambda}) &= \left[\mathcal{O}(\lambda_2, \bar{\lambda}_2) \,,\, \mathcal{O}(\lambda_1, \bar{\lambda}_1)\right] , \\ N(\Lambda) &= \mathcal{O}(\lambda_2, \bar{\lambda}_2) N(\lambda_1) + N(\lambda_2) N(\lambda_1) + \frac{1}{4} M(\lambda_2, \bar{\lambda}_2) M^t(\lambda_1, \bar{\lambda}_1) - \{\lambda_1 \leftrightarrow \lambda_2\} \,, \\ \bar{N}(\bar{\Lambda}) &= \mathcal{O}(\lambda_2, \bar{\lambda}_2) \bar{N}(\bar{\lambda}_1) + \bar{N}(\bar{\lambda}_2) \bar{N}(\bar{\lambda}_1) + \frac{1}{4} M^t(\lambda_2, \bar{\lambda}_2) M(\lambda_1, \bar{\lambda}_1) - \{\lambda_1 \leftrightarrow \lambda_2\} \,. \end{aligned} \tag{A.5}$$

The first equation is straightforward to establish. The second equation can be proven with a bit of algebra. The contribution from the MM^t terms is needed to get the identity to work, confirming that the $[\delta^{(2)}, \delta^{(0)}]$ commutator is crucial to get the gauge algebra to close at this order. The last equation in (A.5) is a consequence of the second and the following discrete symmetry. As we let $D_i \to \bar{D}_i$, $\bar{D}_j \to D_j$, $\lambda_i \to \bar{\lambda}_i$, and $\bar{\lambda}_j \to \lambda_j$ we find that $\mathcal{O}(\lambda, \bar{\lambda})$ is invariant and

$$\Lambda_i \to \bar{\Lambda}_i \,, \quad \bar{\Lambda}_j \to \Lambda_j \,, \quad N(\lambda) \to \bar{N}(\bar{\lambda}) \,, \quad M(\lambda, \bar{\lambda}) \to M^t(\lambda, \bar{\lambda}) \,. \tag{A.6}$$

The third condition in (A.3) guarantees that the terms quadratic in fields work out. A little calculation shows that this condition requires

$$M^t(\Lambda, \bar{\Lambda}) \;=\; \mathcal{O}(\lambda_2, \bar{\lambda}_2) M^t(\lambda_1, \bar{\lambda}_1) + \bar{N}(\bar{\lambda}_2) M^t(\lambda_1) + M^t(\lambda_2, \bar{\lambda}_2) N(\lambda_1) - \{\lambda_1 \leftrightarrow \lambda_2\} \,. \tag{A.7}$$

This actually holds on account of (A.4) and the discrete symmetry. The last equation in (A.3) is needed for the commutator algebra not to acquire cubic terms in the fields. It is quickly confirmed. This concludes our verification of the algebra.

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